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The L_1 Norm of the Approximation Error for Splines with Equidistant Knots*

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1. INTRODUCTION AND STATEMENT OF RESULTS

This paper is concerned with the estimation of the L_1 norm of the difference between a function f of bounded variation in [0, 1] and the associated variation-diminishing spline function with equidistant knots, $S_{m,n}f$, with $n \geqslant m \geqslant 2$; see Schoenberg [4]. For f bounded on [0, 1] and for integers m, n such that

$$n \geqslant m \geqslant 2, \tag{1.1}$$

the function $S_{m,n}f$ is defined by

$$S_{m,n}f(x) = \sum_{j=0}^{l} f(n^{-1}\xi(m,j))\, \tilde{N}_{m,j}(nx), \tag{1.2}$$

where

$$l = m + n - 2, \tag{1.3}$$

$$\xi(m,j) = \begin{cases} \dfrac{(j+1)j}{2(m-1)}, & j = 0,\dots, m-2, \\ j + 1 - \dfrac{m}{2}, & j = m-1,\dots, n-1, \\ n - \xi(m, l-j), & j = n,\dots, l, \end{cases} \tag{1.4}$$

$$\tilde{N}_{m,j}(x) = \begin{cases} \dfrac{j+1}{m} h_{m,j+1}(x), & j = 0,\dots, m-2, \\ h_m(x + m - j - 1), & j = m-1,\dots, n-1, \\ \tilde{N}_{m,l-j}(n - x), & j = n,\dots, l, \end{cases} \tag{1.5}$$

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$$h_{m,k}(x) = \frac{m}{k!} \sum_{i=1}^{k} (-1)^{k-i} i^{k-m} \binom{k}{i} (i-x)_+^{m-1} \quad \text{for} \quad x \geq 0,$$
$$= 0 \quad \text{for} \quad x < 0 \quad (k = 1,..., m). \tag{1.6}$$

$$h_m(x) = h_{m,m}(x). \tag{1.7}$$

In (1.6), y_+ stands for $\max(y, 0)$. (Translated into the notation of Schoenberg [4], $\xi(m,j) = \xi_j$; $h_{m,j+1}(x) = M_j(x)$ for $j = 0,..., m-2$; $h_m(x+m-j-1) = M_j(x)$ for $j = m-1,..., n-1$. The $M_j(nx)$ are the fundamental splines or B-splines for the case of equidistant knots i/n, $i = 1,..., n-1$.)

The total variation of f in [0, 1] will be denoted by $\text{var}(f)$. The sign $\int$ will stand for $\int_{-\infty}^{\infty}$.

THEOREM 1. *If f is of bounded variation in* [0, 1], *then*

$$\int_0^1 | S_{m,n}f(x) - f(x)| \, dx \leq n^{-1} C_m \, \text{var}(f), \tag{1.8}$$

where

$$C_m = \int \left| x - \frac{m}{2} \right| h_{m-1}(x) \, dx. \tag{1.9}$$

The sign of equality holds in (1.8) *if f is a step function whose saltuses are located at $a_i = (k_i - (m/2))/n$, $i = 1,..., r$, where $k_1,..., k_r$ are integers such that $k_1 \geq m$, $k_{i+1} - k_i \geq m$, $i = 1,..., r-1$, $k_r \leq n$, and either* $f(x-) \leq f(x) \leq f(x+)$ or $f(x-) \geq f(x) \geq f(x+)$ for $0 < x < 1$.

The constants C_m satisfy

$$C_m \leq \left(\frac{m+2}{12}\right)^{1/2}, \quad m \geq 2, \tag{1.10}$$

$$C_m = (6\pi)^{-1/2} m^{1/2} + o(m^{1/2}) \quad \text{as} \quad m \to \infty. \tag{1.11}$$

The following theorem shows that if f is a fixed step function satisfying some restrictions, then the upper bound in (1.8) is nearly attained if m and n/m are sufficiently large.

THEOREM 2. *Let* $0 = a_0 < a_1 < \cdots < a_r < a_{r+1} = 1$,

$$f(x) = b_i \quad \text{if} \quad a_{i-1} < x < a_i, \quad i = 1,..., r+1, \tag{1.12}$$

$$f(0) = b_1, \quad f(1) = b_{r+1}, \tag{1.13}$$

$$f(x-) \leq f(x) \leq f(x+) \quad \text{or} \quad f(x-) \geq f(x) \geq f(x+), \quad 0 < x < 1. \tag{1.14}$$

Then if

$$\frac{m}{n} \leqslant \min\{2a_1, a_2 - a_1, \ldots, a_r - a_{r-1}, 2(1 - a_r)\}, \tag{1.15}$$

we have

$$\int_0^1 | S_{m,n}f(x) - f(x)| \, dx$$

$$= \frac{1}{n} \sum_{i=1}^{r} | b_{i+1} - b_i | \int \left| x - \frac{m}{2} + d_i \right| h_{m-1}(x) \, dx \tag{1.16}$$

$$\geqslant \frac{1}{n} \operatorname{var}(f) \int \left| x - \frac{m-1}{2} \right| h_{m-1}(x) \, dx \tag{1.17}$$

$$= \frac{1}{n} \operatorname{var}(f) \, C_m(1 + O(m^{-1})) \qquad \text{as} \quad m \to \infty, \tag{1.18}$$

where d_i is the fractional part of $na_i + m/2$,

$$d_i = na_i + \frac{m}{2} - \left[na_i + \frac{m}{2}\right]. \tag{1.19}$$

Theorems 1 and 2 are proved in Sections 3 and 4, respectively. In Section 2, some needed properties of the fundamental splines $h_{m,k}$ are derived, and a probabilistic interpretation of $h_{m,k}$ is mentioned. Extensions to L_p norm are considered in Section 5. In the Appendix, it is shown that if A belongs to a certain class of linear, constant-preserving operators, then the supremum of the ratio $\int_0^1 | Af - f | \, dx/\operatorname{var}(f)$ over the functions f of (positive) bounded variation is equal to its supremum over the one-step functions.

Theorems 1 and 2 are similar to the results proved in [3] for the Bernstein polynomials,

$$B_nf(x) = \sum_{i=0}^{n} f(i/n) \binom{n}{i} x^i(1 - x)^{n-i}, \qquad 0 \leqslant x \leqslant 1.$$

Theorem 3 of [3] implies that if f is of bounded variation in [0, 1], then

$$\int_0^1 | B_nf(x) - f(x)| \, dx \leqslant (2/e)^{1/2} \, n^{-1/2} J(f) + n^{-1} \operatorname{var}(f), \tag{1.20}$$

where $J(f) = \int_0^1 x^{1/2}(1 - x)^{1/2} \, | df(x)|$. By Theorem 4 of [3], if f is a step function of bounded variation in [0, 1] with finitely many steps in every closed subinterval of (0, 1), then, as $n \to \infty$,

$$\int_0^1 | B_nf(x) - f(x)| \, dx = (2/\pi)^{1/2} \, n^{-1/2} J(f) + o(n^{-1/2}). \tag{1.21}$$

A comparison of (1.21) with Theorem 2 shows that, so far as the approximation of step functions is concerned, the functional $J(f)$ plays the same role in the case of Bernstein polynomials as $\mathrm{var}(f)$ does in the case of the splines $S_{m,n}$. [Note that $J(f) \leqslant \frac{1}{2}\,\mathrm{var}(f)$.] Furthermore, the upper bound on the L_1 norm of the approximation error is of order $n^{-1}m^{1/2}$ in the case of $S_{m,n}$, and of order $n^{-1/2}$ in the case of B_n. Since $S_{m,n}f$ depends on $m + n - 1$ values of f, and $B_n f$ on $n + 1$ values, the approximation by $S_{m,n}f$ is comparable to that by $B_{m+n-2}f$. As $m + n \to \infty$, the ratio, $n^{-1}m^{1/2}$ divided by $(m + n - 2)^{-1/2}$, is asymptotically equal to $(m/n)^{1/2}(1 + m/n)^{1/2}$. Thus insofar as our bounds are indicative of the goodness of approximation, it is more favorable to approximate f by the spline $S_{m,n}f$ with m/n small than by the comparable Bernstein polynomial. This result is similar to the known facts concerning the sup norm in the case of approximation of continuous functions. According to Popoviciu and Schoenberg, respectively (see [4], in particular, Theorem 10), we have

$$\sup_x |\, B_n f(x) - f(x)| \leqslant \tfrac{3}{2}\omega_f(n^{-1/2}),$$

$$\sup_x |\, S_{m,n}f(x) - f(x)| \leqslant \{(m/12)^{1/2} + 1\}\, \omega_f(n^{-1}),$$

where $\omega_f(\cdot)$ denotes the modulus of continuity of f. If f has a bounded derivative, the two upper bounds are of order $n^{-1/2}$ and $n^{-1}m^{1/2}$, respectively.

2. Some Properties of the Fundamental Splines

In this section, some properties of the functions $h_{m,k}$ are derived which will be used in the proofs of the theorems. It is well-known ([1], [4]) that $h_{m,k}$ is a probability density concentrated on the interval $(0, k)$:

$$h_{m,k}(x) \geqslant 0, \quad h_{m,k}(x) = 0 \quad \text{unless } 0 < x < k, \quad \int_{-\infty}^{\infty} h_{m,k}(x)\, dx = 1; \tag{2.1}$$

and that

$$h_m(x) = h_m(m - x), \tag{2.2}$$

$$\sum_{j=0}^{l} \tilde{N}_{m,j}(x) = 1. \tag{2.3}$$

Let

$$H_{m,k}(x) = \int_{-\infty}^{x} h_{m,k}(y)\, dy, \qquad H_m(x) = H_{m,m}(x) = \int_{-\infty}^{x} h_m(y)\, dy. \tag{2.4}$$

From (1.6) we obtain

$$H_{m,k}(x) = 1 - \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} i^{k-m} \binom{k}{i} (i - x)_+^m$$
$$\text{for} \quad x \geqslant 0, \quad k = 1,\ldots, m. \tag{2.5}$$

Let also, for $1 \leqslant k \leqslant m$,

$$G_{m,k}(x) = \frac{m!}{(k-1)!\,(m-k)!} \int_0^x t^{k-1}(1-t)^{m-k}\,dt, \qquad 0 \leqslant x \leqslant 1. \tag{2.6}$$

LEMMA 1. *The following identities hold true*:

$$\frac{k}{m} h_{m,k}(x) = H_{m-1,k-1}(x) - H_{m-1,k}(x), \qquad k = 2,\ldots, m-1, \tag{2.7}$$

$$h_m(x) = H_{m-1}(x) - H_{m-1}(x-1), \tag{2.8}$$

$$H_{m,k}(x) = \int_0^1 H_k(x/t)\,dG_{m,k+1}(t), \qquad k = 1,\ldots, m-1, \tag{2.9}$$

$$\sum_{j=k+1}^{l} \tilde{N}_{m,j}(x) = 1 - \sum_{j=0}^{k} \tilde{N}_{m,j}(x)$$
$$= \begin{cases} H_{m-1,k+1}(x), & k = 0,\ldots, m-2, \\ H_{m-1}(x-k-2+m), & k = m-1,\ldots, n-1, \\ 1 - H_{m-1,l-k}(n-x), & k = n,\ldots, l-1. \end{cases} \tag{2.10}$$

Remark. Identity (2.8) is well-known; the others may be new. Relations (2.7) and (2.10) can be extended to the case of nonequidistant knots by using the results of Greville in the supplement to reference [4]. The second identity (2.10) is analogous to the well-known relation between the binomial and the beta distribution functions, $\sum_{j=k}^{m} \binom{m}{j} x^j(1-x)^{m-j} = G_{m,k}(x)$.

Proof. Since $\binom{k}{i} = \binom{k-1}{i} + \binom{k-1}{i-1}$, we obtain from (1.6) for $k = 2,\ldots, m$ and $x \geqslant 0$

$$\frac{k}{m} h_{m,k}(x) = -\frac{1}{(k-1)!} \sum_{i=1}^{k-1} (-1)^{k-1-i} i^{k-m} \binom{k-1}{i} (i-x)_+^{m-1}$$
$$+ \frac{1}{(k-1)!} \sum_{i=1}^{k} (-1)^{k-i} i^{k-m} \binom{k-1}{i-1} (i-x)_+^{m-1}.$$

The first of the two terms on the right is equal to $H_{m-1,k-1}(x) - 1$ by (2.5). Since $\binom{k-1}{i-1} = i/k\binom{k}{i}$, the second term is equal to $1 - H_{m-1,k}(x)$ for $k \leqslant m-1$, and (2.7) follows. For $k = m$ it is equal to $1 - H_{m-1}(x-1)$, and (2.8) follows.

To prove (2.9) we note that by (2.5) with $m = k$,

$$\int_0^1 H_k(x/t)\, dG_{m,k+1}(t) = 1 - \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} \int_0^1 (i - xt^{-1})_+^k\, dG_{m,k+1}(t).$$

For $x \geqslant i$ the last integral is zero, and for $0 < x < i$,

$$\int_0^1 (i - xt^{-1})_+^k\, dG_{m,k+1}(t) = i^k \frac{m!}{k!\,(m-k-1)!} \int_{x/i}^1 \left(t - \frac{x}{i}\right)^k (1-t)^{m-k-1}\, dt$$
$$= i^k \left(1 - \frac{x}{i}\right)^m.$$

The last equality is obtained by the substitution $s = (t - x/i)/(1 - x/i)$ and evaluation of a beta integral. Identity (2.9) now follows from (2.5).

The first equality in (2.10) is due to (2.3). From (1.5), (2.7) and the easily verified identity

$$\frac{1}{m} h_{m,1}(x) = 1 - H_{m-1,1}(x),$$

we obtain for $k = 0,..., m-2$

$$\sum_{j=0}^{k} N_{m,j}(x) = \sum_{j=0}^{k} \frac{j+1}{m} h_{m,j+1}(x) = 1 - H_{m-1,k+1}(x),$$

which proves the first part of the second equality in (2.10). In particular, $\sum_{j=0}^{m-2} \tilde{N}_{m,j}(x) = 1 - H_{m-1}(x)$. The remaining parts of (2.10) follow from (2.8) and (1.5).

It is of interest to note the following probability interpretation of the fundamental splines $h_{m,k}$. Let $U_1, U_2,...$ be mutually independent random variables, uniformly distributed on (0, 1), and let $U_{m,1} \leqslant U_{m,2} \leqslant \cdots \leqslant U_{m,m}$ denote the ordered $U_1, U_2,..., U_m$. Then $h_{m,k}$ is the probability density of $S_{m,k} = U_{m,1} + \cdots + U_{m,k}$, the sum of the k smallest among $U_1,..., U_m$. Thus $H_{m,k}(x) = \Pr\{S_{m,k} < x\}$. It is known that $G_{m,k}(x) = \Pr\{U_{m,k} < x\}$. Identity (2.9) expresses the fact that, for $1 \leqslant k \leqslant m-1$, $S_{m,k}$ is distributed as the product $S_{k,k} U_{m,k+1}$, where $S_{k,k}$ and $U_{m,k+1}$ are mutually independent with respective distribution functions H_k and $G_{m,k+1}$.

3. Proof of Theorem 1

The operator $S_{m,n}$ is linear, preserves constants [due to (2.3)], and $S_{m,n}f$ depends on f only through values of f on a finite subset of [0, 1]. It follows from Theorem A in the Appendix that if f is of bounded variation in [0, 1], then

$$\int_0^1 | S_{m,n}f(x) - f(x)| \, dx \leqslant \operatorname{var}(f) \sup_{0\leqslant a<1} \int_0^1 | S_{m,n}u_a(x) - u_a(x)| \, dx, \quad (3.1)$$

where

$$u_a(x) = 0 \quad \text{if} \quad x \leqslant a, \qquad u_a(x) = 1 \quad \text{if} \quad x > a. \quad (3.2)$$

Hence inequality (1.8) will be proved if we show that

$$\sup_{0\leqslant a<1} \int_0^1 | S_{m,n}u_a(x) - u_a(x)| \, dx = n^{-1}C_m \,, \quad (3.3)$$

where $C_m = \int | x - m/2 | \, h_{m-1}(x) \, dx$.

We have

$$\int_0^1 | S_{m,n}u_a(x) - u_a(x)| \, dx = n^{-1} \int_0^n \left| S_{m,n}u_a\left(\frac{x}{n}\right) - u_{an}(x) \right| dx, \quad (3.4)$$

$$S_{m,n}u_a\left(\frac{x}{n}\right) = \sum_{j=k(a)+1}^{l} \tilde{N}_j(x),$$

where $k(a) = k$ if $\xi(m, k) \leqslant na < \xi(m, k + 1)$, $k = 0,..., l - 1$. Thus if $\xi(m, k) \leqslant na < \xi(m, k + 1)$ then, by (2.10),

$$S_{m,n}u_a\left(\frac{x}{n}\right) = \begin{cases} H_{m-1,k+1}(x), & k = 0,..., m - 2, \\ H_{m-1}(x - k - 2 + m), & k = m - 1,..., n - 1, \\ 1 - H_{m-1,l-k}(n - x), & k = n,..., l - 1. \end{cases} \quad (3.5)$$

Let $0 \leqslant na < m/2 \,(= \xi(m, m - 1))$. Then $k(a) \leqslant m - 2$ and, with $k(a) = k$,

$$na < \xi(m, k + 1) = \tfrac{1}{2}(k + 2)(k + 1)/(m - 1) \leqslant k + 1.$$

Hence $u_{an}(x) = 1$ for $x > k + 1$. Since also $H_{m-1,k+1}(x) = 1$ for $x > k + 1$, we have

$$\begin{aligned} n \int_0^1 & | S_{m,n}u_a(x) - u_a(x)| \, dx \\ &= \int_0^{k+1} | H_{m-1,k+1}(x) - u_{na}(x)| \, dx \\ &= \int_0^{na} H_{m-1,k+1}(x) \, dx + \int_{na}^{k+1} \{1 - H_{m-1,k+1}(x)\} \, dx. \end{aligned} \quad (3.6)$$

In the interval $\xi(m, k) < na < \xi(m, k+1)$ [in which $k(a) = k$ is constant], the derivative with respect to a of the right side of (3.6) is increasing. Hence the supremum, for a in that interval, is one of the two values at the endpoints of the interval. Therefore,

$$n \sup_{0 \leqslant na < m/2} \int_0^1 | S_{m,n} u_a - u_a | \, dx = \max\{I_{m,1}, \ldots, I_{m,m-1}, J_{m,1}, \ldots, J_{m,m-1}\}, \quad (3.7)$$

where

$$\begin{aligned} I_{m,k} &= \int | H_{m-1,k}(x) - u_{\xi(m,k-1)}(x) | \, dx, \\ J_{m,k} &= \int | H_{m-1,k}(x) - u_{\xi(m,k)}(x) | \, dx. \end{aligned} \quad (3.8)$$

Next, let $m/2 \leqslant na < n - (m/2)(= \xi(m, n-1))$. Then $m - 1 \leqslant k(a) \leqslant n - 2$ and, with $k(a) = k$,

$$\begin{aligned} k + 2 - m \leqslant k + 1 - \frac{m}{2} &= \xi(m, k) \leqslant na < \xi(m, k+1) \\ &= k + 2 - \frac{m}{2} \leqslant k + 1. \end{aligned}$$

Since $H_{m-1}(x - k - 2 + m) = 0$ for $x < k + 2 - m$ and $=1$ for $x > k + 1$, it follows from (3.4) and (3.5) that

$$\begin{aligned} n \int_0^1 | S_{m,n} u_a - u_a | \, dx &= \int_{k+2-m}^{k+1} | H_{m-1}(x - k - 2 + m) - u_{an}(x) | \, dx \\ &= \int_0^{m-1} | H_{m-1}(y) - u_b(y) | \, dy, \end{aligned} \quad (3.9)$$

where $b = na + m - k - 2$. The interval $\xi(m, k) \leqslant na < \xi(m, k+1)$ corresponds to $m/2 - 1 \leqslant b < m/2$, that is, $\xi(m, m-2) \leqslant b < \xi(m, m-1)$. It follows, as above, that

$$\sup_{m/2 \leqslant na < n - m/2} n \int_0^1 | S_{m,n} u_a - u_a | \, dx = \max(I_{m,m-1}, J_{m,m-1}). \quad (3.10)$$

Finally, it can be seen from (3.5) that the supremum for $n - (m/2) \leqslant na < n$ is equal to the supremum for $0 < na \leqslant m/2$. Hence, and by (3.7) and (3.10),

$$n \sup_{0 \leqslant a < 1} \int_0^1 | S_{m,n} u_a - u_a | \, dx = \max\{I_{m,1}, \ldots, I_{m,m-1}, J_{m,1}, \ldots, J_{m,m-1}\}. \quad (3.11)$$

Referring to (3.8), it is easy to show that

$$\begin{aligned} I_{m,k} &= \int | x - \xi(m, k-1) | \, dH_{m-1,k}(x), \\ J_{m,k} &= \int | x - \xi(m, k) | \, dH_{m-1,k}(x). \end{aligned} \quad (3.12)$$

Due to (2.2),

$$I_{m,m-1} = \int \left| x - \frac{m-2}{2} \right| dH_{m-1}(x)$$

$$= \int \left| x - \frac{m}{2} \right| dH_{m-1}(x) = J_{m,m-1}, \tag{3.13}$$

which, by (1.9), is equal to C_m. Hence, (3.3) will be proved if we show that

$$I_{m,k} \leqslant I_{m,m-1} \quad \text{and} \quad J_{m,k} \leqslant J_{m,m-1}, \quad k = 1,..., m-2. \tag{3.14}$$

From (2.9) we obtain

$$h_{m-1,k}(x) = \int_0^1 h_k(xt^{-1})\, t^{-1}\, dG_{m-1,k+1}(t), \qquad k = 1,..., m-2.$$

Hence, by (3.12),

$$J_{m,k} = \int_{-\infty}^{\infty} | x - \xi(m, k)| \int_0^1 h_k(xt^{-1})\, t^{-1}\, dG_{m-1,k+1}(t)\, dx$$

$$= \int_{-\infty}^{\infty} \int_0^1 | ty - \xi(m, k)|\, dG_{m-1,k+1}(t)\, h_k(y)\, dy$$

$$\geqslant \int_{-\infty}^{\infty} \left| \int_0^1 (ty - \xi(m, k))\, dG_{m-1,k+1}(t) \right| h_k(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left| \frac{k+1}{m} y - \xi(m, k) \right| h_k(y)\, dy.$$

An analogous inequality, with $\xi(m, k)$ replaced by $\xi(m, k-1)$, holds for $I_{m,k}$. Thus we have

$$I_{m,k} \geqslant A_k, \qquad J_{m,k} \geqslant B_k, \qquad k = 1,..., m-2, \tag{3.15}$$

where

$$A_k = \frac{k+1}{m} \int | x - a_k |\, h_k(x)\, dx,$$
$$B_k = \frac{k+1}{m} \int | x - b_k |\, h_k(x)\, dx, \tag{3.16}$$

$$a_k = \frac{m}{k+1}\, \xi(m, k-1), \qquad b_k = \frac{m}{k+1}\, \xi(m, k). \tag{3.17}$$

Note that $A_{m-1} = I_{m,m-1}$ and $B_{m-1} = J_{m,m-1}$. Hence to prove (3.14), it is sufficient to show that

$$A_{k-1} \leqslant A_k \quad \text{and} \quad B_{k-1} \leqslant B_k, \quad k = 2,..., m-1. \tag{3.18}$$

According to (2.8), $h_k(x) = \int_0^1 h_{k-1}(x-t)\,dt$. Hence,

$$\int |x-a_k|\,h_k(x)\,dx = \int |x-a_k| \int_0^1 h_{k-1}(x-t)\,dt\,dx$$
$$= \iint_0^1 |y+t-a_k|\,dt\,h_{k-1}(y)\,dy$$
$$\geqslant \int \left| \int_0^1 (y+t-a_k)\,dt \right| h_{k-1}(y)\,dy$$
$$= \int |y+\tfrac{1}{2}-a_k|\,h_{k-1}(y)\,dy.$$

Since $h_{k-1}(y) = h_{k-1}(k-1-y)$,

$$\int |y+\tfrac{1}{2}-a_k|\,h_{k-1}(y)\,dy$$
$$= \int |k-\tfrac{1}{2}-a_k-y|\,h_{k-1}(y)\,dy$$
$$= \frac{1}{2}\int (|y+\tfrac{1}{2}-a_k| + |y+\tfrac{1}{2}+a_k-k|)\,h_{k-1}(y)\,dy$$
$$= \int \max\left(\left|y-\frac{k-1}{2}\right|, \left|a_k-\frac{k}{2}\right|\right) h_{k-1}(y)\,dy. \tag{3.19}$$

Hence,

$$A_k \geqslant \frac{k+1}{m}\int \max\left(\left|x-\frac{k-1}{2}\right|, \left|a_k-\frac{k}{2}\right|\right) h_{k-1}(x)\,dx. \tag{3.20}$$

On the other hand, by (3.19) with $a_k - \frac{1}{2}$ replaced by a_{k-1} and (3.16),

$$A_{k-1} = \frac{k}{m}\int \max\left(\left|x-\frac{k-1}{2}\right|, \left|a_{k-1}-\frac{k-1}{2}\right|\right) h_{k-1}(x)\,dx. \tag{3.21}$$

Hence the first inequalities (3.18) are satisfied if

$$k\left|a_{k-1}-\frac{k-1}{2}\right| \leqslant (k+1)\left|a_k-\frac{k}{2}\right|, \qquad k = 2,\ldots,m-1,$$

or equivalently, if

$$\left|\xi(m,k-2)-\frac{1}{m}\binom{k}{2}\right| \leqslant \left|\xi(m,k-1)-\frac{1}{m}\binom{k+1}{2}\right|,$$
$$k = 2,\ldots,m-1. \tag{3.22}$$

Now

$$m\xi(m, k-1) - \binom{k+1}{2}$$
$$= k\,\frac{m(k-1)-(m-1)(k+1)}{2(m-1)} = k\,\frac{-2m+k+1}{2(m-1)} < 0$$

for $1 \leqslant k \leqslant m-1$, hence,

$$\left|\,\xi(m, k-1) - \frac{1}{m}\binom{k+1}{2}\right| - \left|\,\xi(m, k-2) - \frac{1}{m}\binom{k}{2}\right|$$
$$= \frac{k(2m-k-1)-(k-1)(2m-k)}{2m(m-1)} = \frac{m-k}{m(m-1)} > 0$$

for $2 \leqslant k \leqslant m-1$, and (3.22) holds true.

Replacing a_k by b_k, we see that the second inequalities (3.18) are satisfied if

$$\left|\,\xi(m, k-1) - \frac{1}{m}\binom{k}{2}\right| \leqslant \left|\,\xi(m, k) - \frac{1}{m}\binom{k+1}{2}\right|,$$
$$k = 2, \ldots, m-1. \qquad (3.23)$$

This is true since $\xi(m, k) - \binom{k+1}{2}/m = \binom{k+1}{2}/m(m-1)$. This completes the proof of (3.3) and thus of inequality (1.8).

The stated condition for equality in (1.8) follows from Theorem 2, in particular (1.16) with $d_i = 0$ for all i.

To prove inequality (1.10), we observe that

$$C_m = \int \left| x - \frac{m}{2} \right| dH_{m-1}(x) \leqslant \left\{ \int \left(x - \frac{m}{2} \right)^2 dH_{m-1}(x) \right\}^{1/2}.$$

Since H_{m-1} is the distribution function of the sum of $m-1$ independent random variables whose mean and variance are 1/2 and 1/12, respectively, we have

$$\int x\, dH_{m-1}(x) = \frac{m-1}{2}, \qquad \int \left(x - \frac{m-1}{2} \right)^2 dH_{m-1}(x) = \frac{m-1}{12}.$$

Therefore, $\int (x - m/2)^2\, dH_{m-1} = (m-1)/12 + 1/4 = (m+2)/12$, and (1.10) follows.

Finally, to prove (1.11) we write

$$C_m = \int_0^{m-1} |\, H_{m-1}(x) - u_{m/2}(x) |\, dx$$
$$= \int_0^{m/2} H_{m-1}(x)\, dx + \int_0^{(m-2)/2} H_{m-1}(x)\, dx.$$

Hence as $m \to \infty$,

$$C_m = 2 \int_0^{(m-1)/2} H_{m-1}(x)\, dx + O(1). \tag{3.24}$$

Let

$$H_{m-1}^*(x) = H_{m-1}\left(\frac{m-1}{2} + x\left(\frac{m-1}{12}\right)^{1/2}\right).$$

Referring to the preceding paragraph, it follows from the central limit theorem (see, e.g., [2], p. 253) that

$$\lim_{m\to\infty} H_{m-1}^*(x) = \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-t^2/2)\, dt \tag{3.25}$$

uniformly for $-\infty < x < \infty$.

Now

$$\int_0^{(m-1)/2} H_{m-1}(x)\, dx = \left(\frac{m-1}{12}\right)^{1/2} \int_{-\infty}^{0} H_{m-1}^*(x)\, dx. \tag{3.26}$$

For $a \in (-\infty, 0)$ fixed, $\int_a^0 H_{m-1}^*(x)\, dx \to \int_a^0 \Phi(x)\, dx$ as $m \to \infty$. By Chebyshev's inequality, $H_{m-1}^*(x) \leqslant |x|^{-2}$ for $x < 0$, hence $\int_{-\infty}^{a} H_{m-1}^*(x)\, dx \leqslant |a|^{-1}$ for $a < 0$. Therefore

$$\lim_{m\to\infty} \int_{-\infty}^{0} H_{m-1}^*(x)\, dx = \int_{-\infty}^{0} \Phi(x)\, dx = -\int_{-\infty}^{0} y\Phi'(y)\, dy$$

$$= \int_{-\infty}^{0} \Phi''(y)\, dy = \Phi'(0) = (2\pi)^{-1/2}. \tag{3.27}$$

Equality (1.11) now follows from (3.24), (3.26), and (3.27). Theorem 1 is proved.

4. Proof of Theorem 2

A step function f which satisfies (1.12) and (1.13) may be written

$$f(x) = f(0) + \sum_{i=1}^{r} (b_{i+1} - f(a_i))\, u_{a_i}(x) + \sum_{i=1}^{r} (f(a_i) - b_i)\, u_{a_i}^{+}(x),$$

where $u_a(x)$ is defined in (3.2) and

$$u_a{}^{+}(x) = u_a(x+) = 0 \text{ or } 1 \text{ according as } x < a \text{ or } x \geqslant a.$$

Hence,

$$S_{m,n}f(x) - f(x) = \sum_{i=1}^{r} (b_{i+1} - f(a_i))\{S_{m,n}u_{a_i}(x) - u_{a_i}(x)\}$$
$$+ \sum_{i=1}^{r} (f(a_i) - b_i)\{S_{m,n}u_{a_i}^{+}(x) - u_{a_i}^{+}(x)\}. \quad (4.1)$$

Let

$$k_i = \left[na_i + \frac{m}{2}\right], \qquad i = 1,..., r.$$

By (1.19), $k_i = na_i + m/2 - d_i$, and since $0 \leqslant d_i < 1$, condition (1.15) implies

$$k_1 \geqslant m, \qquad k_i - k_{i-1} \geqslant m, \qquad i = 2,..., r; \quad k_r \leqslant n. \quad (4.2)$$

By (3.5),

$$S_{m,n}u_{a_i}(x) = H_{m-1}(nx - k_i - 1 + m) \qquad \text{if} \quad m - 1 \leqslant k_i \leqslant n. \quad (4.3)$$

Also, $S_{m,n}u_{a_i}^{+}(x) = S_{m,n}u_{a_i}(x)$ unless $na_i + m/2$ is an integer, in which case

$$S_{m,n}u_{a_i}^{+}(x) = H_{m-1}(nx - k_i + m) \qquad \text{if} \quad m \leqslant k_i \leqslant n + 1. \quad (4.4)$$

It follows that if $nx < k_i - m$ or $nx > k_i$, then

$$S_{m,n}u_{a_i}(x) - u_{a_i}(x) = 0 \qquad \text{and} \qquad S_{m,n}u_{a_i}^{+}(x) - S_{m,n}u_{a_i}^{+}(x) = 0.$$

Hence, and due to (4.2),

$$S_{m,n}f(x) - f(x) = (b_{i+1} - f(a_i))\{S_{m,n}u_{a_i}(x) - u_{a_i}(x)\}$$
$$+ (f(a_i) - b_i)\{S_{m,n}u_{a_i}^{+}(x) - u_{a_i}^{+}(x)\}$$
$$\text{if} \quad k_{i-1} < nx < k_i, \quad i = 1,..., r, \quad (4.5)$$

where $k_0 = 0$; and $S_{m,n}f(x) - f(x) = 0$ for $nx > k_r$.

Due to condition (1.14), the two terms on the right of (4.5) are of the same sign (except at $x = a_i$). Therefore,

$$\int_0^1 | S_{m,n}f(x) - f(x)| \, dx = \sum_{i=1}^{r} \left\{| b_{i+1} - f(a_i)| \int_{k_{i-1}}^{k_i} | S_{m,n}u_{a_i}(x) - u_{a_i}(x)| \, dx\right\}$$
$$+ | f(a_i) - b_i | \int_{k_{i-1}}^{k_i} | S_{m,n}u_{a_i}^{+}(x) - u_{a_i}^{+}(x)| \, dx.$$

The limits of integration, k_{i-1} and k_i, may be replaced by $-\infty$ and ∞. For the first integral in the ith term on the right we obtain, using (4.3), as in the proof of Theorem 1,

$$\int | S_{m,n}u_{a_i}(x) - u_{a_i}(x)| \, dx = n^{-1} \int \left| x - \frac{m}{2} + 1 - d_i \right| dH_{m-1}(x).$$

Due to the symmetry of h_{m-1}, this is also equal to

$$n^{-1} \int \left| x - \frac{m}{2} + d_i \right| dH_{m-1}(x).$$

The second integral can be seen to have the same value. [Note that formula (4.4) applies only when $d_i = 0$.] Since by (1.14),

$$| b_{i+1} - f(a_i)| + |f(a_i) - b_i | = | b_{i+1} - b_i |,$$

this implies (1.16).

Inequality (1.17) follows from the well-known fact that if $H(x)$ is a probability distribution function and $H(c) = 1/2$, then $\int | x - y | \, dH(x)$ is minimized at $y = c$. Note that $\operatorname{var}(f) = \sum_{i=1}^{r} | b_{i+1} - b_i |$.

Finally,

$$\begin{aligned} C_m &- \int \left| x - \frac{m-1}{2} \right| h_{m-1}(x) \, dx \\ &= \int \left(\left| x - \frac{m}{2} \right| - \left| x - \frac{m-1}{2} \right| \right) h_{m-1}(x) \, dx \\ &= \int_{m/2-1}^{m/2} \left(\frac{1}{2} - \left| x - \frac{m-1}{2} \right| \right) h_{m-1}(x) \, dx \\ &\leqslant \frac{1}{2} \left\{ H_{m-1}\left(\frac{m}{2}\right) - H_{m-1}\left(\frac{m}{2} - 1\right) \right\} = \frac{1}{2} h_m\left(\frac{m}{2}\right). \end{aligned}$$

By the central limit theorem for densities (see, e.g., [2], p. 489), $h_m(m/2) = O(m^{-1/2})$. Since, by (1.11), C_m is asymptotically proportional to $m^{1/2}$, these facts imply (1.18). Theorem 2 is proved.

5. Extension to L_p norm, $p > 1$

Theorem 1 can be extended as follows: If f is of bounded variation in $[0, 1]$, then for $p \geqslant 1$

$$\left\{ \int_0^1 | S_{m,n}f(x) - f(x)|^p \, dx \right\}^{1/p} \leqslant n^{-1/p} C_m(p) \operatorname{var}(f), \tag{5.1}$$

where,

$$C_m(p) = \max\{I_{m,1}(p),\dots, I_{m,m-1}(p), J_{m,1}(p),\dots, J_{m,m-1}(p)\}, \tag{5.2}$$

$$\begin{aligned} I_{m,k}(p)^p &= \int | H_{m-1,k}(x) - u_{\xi(m,k-1)}(x)|^p \, dx, \\ J_{m,k}(p)^p &= \int | H_{m-1,k}(x) - u_{\xi(m,k)}(x)|^p \, dx. \end{aligned} \tag{5.3}$$

The constants $C_m(p)$ satisfy

$$C_m(p) \leqslant \left(\frac{m+2}{12}\right)^{1/(2p)}, \qquad m \geqslant 2, \tag{5.4}$$

$$C_m(p)^p = 3^{-1/2} \int_{-\infty}^{0} \Phi(x)^p \, dx \, m^{1/2} + o(m^{1/2}) \qquad \text{as} \quad m \to \infty, \tag{5.5}$$

where $\Phi(x)$ is defined in (3.25).

The proof of (5.1) is essentially the same as that of the corresponding part of Theorem 1. (See Remark 2 at the end of the Appendix.) Inequality (5.4) follows from (1.10) since $C_m(p)^p \leqslant C_m(1) = C_m$ for $p > 1$. However, (5.5) with $p > 1$ cannot be proved in the same way as (1.11). The reason is that for $p > 1$, the maxima of $I_{m,k}(p)$ and $J_{m,k}(p)$ with respect to $k(1 \leqslant k \leqslant m-1)$ are not, in general, attained at $k = m - 1$ [although (5.5) implies that this is true in an asymptotic sense as $m \to \infty$]. The main steps in the proof of (5.5) are as follows.

Let $\mu(m, k)$ and $\sigma^2(m, k)$ denote the mean and the variance of the distribution $H_{m,k}$, and let $H^*_{m,k}(x) = H_{m,k}(\mu(m, k) + \sigma(m, k)x)$. The following is crucial:

LEMMA 5.1. *For $\delta \in (0, 1)$ arbitrarily fixed,*

$$\lim_{m\to\infty} H^*_{m,k}(x) = \Phi(x)$$

uniformly in $x(-\infty < x < \infty)$ and uniformly in k for $\delta m < k \leqslant m$.

The proof of Lemma 5.1 makes use of the representation (2.9) and of the asymptotic normality of the distributions H_k and $G_{m,k+1}$.

It is easy to show that $C_m(p)$ behaves asymptotically as $\max\{J_{m,1}(p),\dots, J_{m,m-1}(p)\}$, and that for $k \leqslant \delta m$, $J_{m,k}(p)^p \leqslant J_{m,k}(1) \leqslant C\delta m^{1/2}$, where C is an absolute constant. Hence, with $\delta > 0$ suitably fixed, the maximum of $J_{m,k}(p)$ may be taken over the range $\delta m < k \leqslant m - 1$. With the help of Lemma 5.1, it can be shown that asymptotically as $m \to \infty$,

$$J_{m,k}(p)^p \sim \left\{\frac{m}{3}\left(\frac{k}{m}\right)^3\left(4 - 3\frac{k}{m}\right)\right\}^{1/2} \int_{-\infty}^{0} \Phi(x)^p \, dx,$$

uniformly for $\delta m < k < m$. The right-hand side increases with k for $k \leqslant m$, and (5.5) follows.

There is a straightforward but uninteresting extension of Theorem 2 to the case $p > 1$. Under the conditions of Theorem 2, as $m \to \infty$, the asymptotic value of the left-hand side of (5.1) differs from that of the right-hand side in that $\operatorname{var}(f) = \sum_{i=1}^{r} | b_{i+1} - b_i |$ is replaced by $\{\sum_{i=1}^{r} | b_{i+1} - b_i |^p\}^{1/p}$. Thus if $p > 1$, we do not have asymptotic equality in (5.1) for a class of step functions as extensive as that of Theorem 2.

Appendix

Let BV and L_1 denote the classes of functions of bounded variation in [0, 1] and of Lebesgue integrable functions on [0, 1], respectively. Let $u_a(x) = 0$ or 1 according as $x \leqslant a$ or $x > a$.

THEOREM A. *If* $A\colon BV \to L_1$ *is a linear, constant-preserving operator, and* Af *depends on* f *only through values of* f *on a finite subset of* [0, 1], *then for every* $f \in BV$

$$\int_0^1 | Af(x) - f(x)| \, dx \leqslant \operatorname{var}(f) \sup_{0 \leqslant a < 1} \int_0^1 | Au_a(x) - u_a(x)| \, dx. \quad \text{(A.1)}$$

Proof. Let Af depend on f only through $f(a_i)$, $i = 1, \ldots, m$, where $0 \leqslant a_i \leqslant 1$. Given $\epsilon > 0$, choose $n \geqslant m$ and $0 = b_0 < b_1 < \cdots < b_{n-1} < b_n = 1$ so that the numbers a_i are among the b_j and $\max_j(b_{j+1} - b_j) \leqslant \epsilon$.

First, let f be a nondecreasing bounded function on [0, 1]. Define the function f_0 by

$$f_0(0) = f(0), \qquad f_0(x) = f(b_j) \quad \text{if} \quad b_{j-1} < x \leqslant b_j, \qquad j = 1, \ldots, n.$$

Since $f_0(a_i) = f(a_i)$ for all i, we have $Af_0 = Af$. Also,

$$\int_{b_{j-1}}^{b_j} | f - f_0 | \, dx \leqslant \epsilon(f(b_j) - f(b_{j-1})),$$

so that $\int_0^1 | f - f_0 | \, dx \leqslant \epsilon(f(1) - f(0)) = \epsilon \operatorname{var}(f)$. It follows that

$$\int_0^1 | Af - f | \, dx = \int_0^1 | Af_0 - f | \, dx \leqslant \int_0^1 | Af_0 - f_0 | \, dx + \epsilon \operatorname{var}(f). \quad \text{(A.2)}$$

We can write $f_0(x) = f(0) + \sum_{j=0}^{n-1} c_j u_{b_j}(x)$, $0 \leqslant x \leqslant 1$, where $c_j =$

$f(b_{j+1}) - f(b_j) \geqslant 0$. Since the operator A is linear and preserves constants, we have $Af_0 - f_0 = \sum_{j=0}^{n-1} c_j(Au_{b_j} - u_{b_j})$. Therefore,

$$\int_0^1 | Af_0 - f_0 | \, dx \leqslant \sum_{j=0}^{n-1} c_j \int | Au_{b_j} - u_{b_j} | \leqslant \sum_{j=0}^{n-1} c_j \sup_{0 \leqslant a < 1} \int | Au_a - u_a | \, dx$$

$$= \operatorname{var}(f) \sup_{0 \leqslant a < 1} \int | Au_a - u_a | \, dx. \tag{A.3}$$

Since ϵ is arbitrary, inequality (A.1), for f nondecreasing, follows from (A.2) and (A.3).

A function f of bounded variation on [0, 1] can be represented as $f = f_1 - f_2$, where f_1 and f_2 are nondecreasing and $\operatorname{var}(f) = \operatorname{var}(f_1) + \operatorname{var}(f_2)$. Since the operator A is linear, $Af - f = (Af_1 - f_1) - (Af_2 - f_2)$, implying

$$\int_0^1 | Af - f | \, dx \leqslant \int_0^1 | Af_1 - f_1 | \, dx + \int_0^1 | Af_2 - f_2 | \, dx,$$

and since (A.1) is true with f replaced by f_1 and f_2, it is true in the general case.

Remark 1. The assumption that Af depends on f only through values of f on a finite set is clearly not essential. For instance, it can be replaced by the assumption that $\int | Af | \, dx \leqslant M \int | f | \, dx$ for $f \in BV$.

Remark 2. Theorem A is also true with L_1 norm replaced by L_p norm, $p > 1$. The proof is virtually the same and uses Minkowski's inequality.

Remark 3. Under the condition of Theorem A we have the following counterpart to inequality (A.1):

$$\left| \int_0^1 (Af(x) - f(x)) \, dx \right| \leqslant \operatorname{var}(f) \sup_{0 \leqslant a < 1} \left| \int_0^1 (Au_a(x) - u_a(x)) \, dx \right|. \tag{A.4}$$

The proof is similar to that of (A.1). The special case of (A.4) with $Af(x) = f(y_{[Nx]+1})$, where $y_1, \ldots, y_N$ is a sequence in [0, 1), is due to J. F. Koksma [5], except that in Koksma's inequality the u_a in (A.4) are replaced by the indicator functions of the subintervals $[\alpha, \beta)$ of [0, 1]. Koksma's inequality has been extended by Hlawka [6] to the multidimensional case; see also Zaremba [7]. The extension of inequalities (A.1) and (A.4) to the multidimensional case, under conditions analogous to those of Theorem A, is not difficult. I am indebted to Professor Walter Philipp for directing my attention to the references here mentioned.

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